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A Fixed Point Stability Theorem for Nonexpansive Set Valued Mappings

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The behavior of the fixed points of set valued maps, as these maps vary, has been considered in [6] and [7]. Both papers establish conditions implying the strong convergence of the fixed points of sequences of set valued contractions. Here a similar result is given for weak convergence of the fixed points of set valued maps in a Banach space where the maps are only assumed to be nonexpansive, i.e., $D(F(x), F(y)) \leqslant \| x - y \|$, where D is the Hausdorff metric generated by the norm $\| \cdot \|$. This theorem is applied to obtain a stability result for generalized differential equations.

1. A map J of a Banach space X into its dual X^* is a duality map if $(x, J(x)) = \| J(x) \| \| x \|$ and $\| J(x) \| = \mu(\| x \|)$ for $x \in X$, where μ is a nonnegative, nondecreasing function on R^1 with $\mu(0) = 0$. The map J is said to be weakly continuous if it is continuous on X with the weak topology into X^* with the weak * topology. Weak convergence of a sequence $\{x_i\}$ to a point x_0 is denoted by $x_i \rightharpoonup x_0$.

A map G of a Banach space X into the nonempty subsets of X is called a J-monotone map, if for each pair $x, y \in X$ and $x_1 \in G(x)$ there is a $y_1 \in G(y)$ for which $(x_1 - y_1, J(x - y)) \geqslant 0$, where J is a duality map on X.

A J-monotone map G is said to be maximal monotone provided: If for a given pair $x, x_1 \in X$ and any $y \in X$ there is a $y_1 \in G(y)$ such that $(x_1 - y_1, J(x - y)) \geqslant 0$, then $x_1 \in G(x)$. A J-monotone map of a strictly convex Banach space with a weakly continuous duality map into $K(X)$, the family of nonempty compact convex subsets of X, is maximal monotone [1]. Under the same assumptions it is also shown in [1] that a nonexpansive map of a weakly compact convex set B into $K(B)$ has a nonempty fixed point set. The strictly convex Banach spaces with a weakly continuous duality map include Hilbert space and the 1_p spaces for $1 < p < \infty$.

THEOREM 1. *Let X be a strictly convex Banach space with a weakly continuous duality map, and let B be a weakly compact convex subset of X. Assume* *that* $\{F_i\}$ *is a sequence of nonexpansive maps of* X *into* $K(X)$ *converging pointwise to* F_0 *in the Hausdorff metric* D *and mapping* B *into itself. If* x_i *is a fixed point of* F_i *in* B *for* $i = 1, 2,\ldots$ *and* $x_i \rightharpoonup x_0$, *then* x_0 *is a fixed point of* F_0.

Proof. For any nonexpansive map F of X into $K(X)$, the map $I - F$ is J-monotone [1]. Thus the maps $I - F_i$, $i = 0, 1,\ldots$, are each J-monotone. Since x_i is a fixed point of F_i, we have $0 \in (I - F_i)(x_i)$, $i = 1, 2,\ldots$. By the J-monotonicity property for any $v \in X$, there is a $v_i \in (I - F_i)(v)$ such that

$$(v_i - 0, J(v - x_i)) \geqslant 0 \tag{1}$$

for $i = 1, 2,\ldots$. The pointwise convergence of the sequence $\{F_i\}$ implies that $\lim_{i\to\infty} D((I - F_i)(v), (I - F_0)(v)) = 0$ for any $v \in X$. Thus, since the sets $(I - F_i)(v)$ are compact for $i = 0, 1,\ldots$, the set $\bigcup_{i=0}^{\infty} (I - F_i)(v)$ is also compact for $v \in X$ [2], and the sequence $\{v_i\}$ can be assumed convergent to a point $v_0 \in (I - F_0)(v)$. Taking the limit in (1) we have

$$(v_0 - 0, J(v - x_0)) \geqslant 0. \tag{2}$$

The map $I - F_0$ is maximal monotone by [1], so that the inequality (2) implies $0 \in (I - F_0)(x_0)$. This establishes that x_0 is a fixed point of F_0.

2. Let B be a closed origin centered ball of radius r in Euclidean n — space R^n. Consider the family of generalized differential equations

$$x'(t) \in R_i(t, x(t)) \tag{3}$$

satisfying the initial condition $x(0) = x_0$ for $i = 0, 1,\ldots$. The right sides of Eqs. (3) are assumed to satisfy the conditions

a. Each R_i is a continuous map of $[0, a] \times R^n$ into $K(R^n)$ and maps $[0, a] \times B$ into $K(B)$.

b. $\operatorname{Lim}_{i\to\infty} D(R_i(t, u), R_0(t, u)) = 0$ for $(t, u) \in [0, a] \times R^n$, where D is the Hausdorff metric generated by the Euclidean norm $\|\cdot\|$.

c. There is a constant $k \geqslant 0$ such that $D(R_i(t, u), R_i(t, v)) \leqslant k \| u - v \|$ for $u, v \in R^n$.

A solution to (3) is any absolutely continuous map y satisfying (3) a.e. Condition a guarantees the existence of a solution for each of the equations (3) on $[0, a]$ by a result of Castaing [3].

Let $C[0, a](C_r[0, a])$ denote the space of continuous maps of $[0, a]$ into R^n (which satisfy a Lipschitz condition with constant r), and let $L_2[0, a]$ denote the square integrable maps of $[0, a]$ into R^n with the usual norm. Define the maps S_i of $L_2[0, a]$ into the subsets of $L_2[0, a]$ by

$$S_i(x) = \left\{ y \in L_2[0, a] : y(t) = x_0 + \int_0^t u(s)\, ds,\ u(s) \in R_i(s, x(s)) \right\}$$

for $i = 0, 1, \ldots$. The values of the S_i are nonempty compact convex subsets of $C[0, a]$ with the sup norm by a result of Bridgeland [4] and hence, are compact convex subsets of $L_2[0, a]$. For $x \in C_r[0, a]$ it is easily seen that $S_i(x) \subseteq C_r[0, a]$ for $i = 0, 1, \ldots$.

Letting H denote the Hausdorff metric generated by the norm of $L_2[0, a]$, it is also shown in [4] that $\lim_{i\to\infty} H(S_i(x), S_0(x)) = 0$ for $x \in L_2[0, a]$. In addition, condition c implies that the maps S_i satisfy the Lipschitz condition

$$H(S_i(x), S_i(y)) \leqslant ka^{1/2} \left(\int_0^a \| x(s) - y(s) \|^2 \, ds\right)^{1/2}$$

for $x, y \in L_2[0, a]$ and $i = 0, 1, \ldots$ [5].

Based upon the above remarks, it is clear that the closed bounded convex subset $C_r[0, a]$ of the Hilbert space $L_2[0, a]$ and the maps $\{S_i\}$, $i = 0, 1, \ldots$, satisfy the conditions of Theorem 1, provided that $ka^{1/2} \leqslant 1$. Observing that the fixed point set of the map S_i is precisely the solution set of equations (3) for $i = 0, 1, \ldots$, we can state

THEOREM 2. *Let the right side of the generalized differential equation* $(*)$ $x'(t) \in R_i(t, x(t))$, $x(0) = x_0$, *satisfy conditions a–c for* $i = 0, 1, \ldots$, *and let* $ka^{1/2} \leqslant 1$. *If* y_i *is a solution to* $(*)$ *for* $i = 1, 2, \ldots$, *and* $y_i \rightharpoonup y_0$ *in* $L_2[0, a]$, *then* y_0 *is a solution of* $x'(t) \in R_0(t, x(t))$, $x(0) = x_0$.

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